



## Single facility collection depots location problem in the plane

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### ABSTRACT

In this paper we consider an extension of the classical facility location problem where besides  $n$  weighted customers, a set of  $p$  collection depots are also given. In this setting the service of a customer consists of the travel of a server to the customer and return back to the center via a collection depot. We have analyzed the problem and showed that the collection depots problem using the Euclidean metric can be transformed to  $O(p^2n^2)$  number of different classical facility location problems and this bound is tight. We then show the existence of small coresets for these problems. These coresets are then used to provide  $(1 + \epsilon)$ -factor approximation algorithms which have linear running times for fixed customer weights and  $\epsilon$ .

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## 1. Introduction

Given is a weighted set of customers or demand points  $C = \{c_1, c_2, \dots, c_n\}$ . The weight  $w_i$  of each customer is assumed positive and constant. Also given is a set of collection depots  $D = \{d_1, d_2, \dots, d_p\}$ . A facility serving a customer dispatches a vehicle that visits the customer and returns to the facility through the collection depot which provides the shortest route. The goal is to minimize the travelled weighted Euclidean distance. The objective function to be minimized depends on the application. One of the widely used objective functions is to locate the facility at the point that minimizes the maximum of the weighted distances of the round trip to all the customers. That is, the goal is to minimize  $F(s)$ , where

$$F(s) = \max_{i=1, \dots, n} w_i \cdot \left\{ \text{dist}(s, c_i) + \min_{j=1, \dots, p} \left\{ \text{dist}(c_i, d_j) + \text{dist}(d_j, s) \right\} \right\} \quad (1)$$

Here  $\text{dist}(a, b)$  indicates the Euclidean distance between the points  $a$  and  $b$ . This problem is known as the 1-center or MinMax collection depots problem. In this paper we will always assume that the distance metric is Euclidean.

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Another objective function is  $G(s)$ , where

$$G(s) = \sum_{i=1}^n w_i \cdot \left\{ \text{dist}(s, c_i) + \min_{j=1, \dots, p} \left\{ \text{dist}(c_i, d_j) + \text{dist}(d_j, s) \right\} \right\} \quad (2)$$

This problem is known as the 1-median or the MinSum collection depots problem.

We see that the depot associated with a customer varies as the service center is moved. Let  $I_s$  denote the assignment vector of length  $n$  where  $I_s[i]$  indicates the depot assigned to customer  $c_i$  when the service facility is at  $s$ . In this case  $F(s)$  and  $G(s)$  can be rewritten as

$$F(s) = \max_{i=1, \dots, n} w_i \cdot \left\{ \text{dist}(s, c_i) + \left\{ \text{dist}(c_i, I_s[i]) + \text{dist}(I_s[i], s) \right\} \right\} \quad (3)$$

$$G(s) = \sum_{i=1}^n w_i \cdot \left\{ \text{dist}(s, c_i) + \left\{ \text{dist}(c_i, I_s[i]) + \text{dist}(I_s[i], s) \right\} \right\} \quad (4)$$

Note that the assignment vector is the same for a particular  $s$  for both the objective functions  $F(s)$  and  $G(s)$ .

The MinMax and MinSum collection depots problems are essentially generalized versions of the classical MinMax and MinSum facility location problems respectively (consider the case where every client also coincides with a collection depot). Different variations of this problem can be defined, depending on the distance metric used. Several applications are described in [6,11], such as septic tank cleaning, garbage collection, and tree pruning services. In each of these applications, it may be desirable to minimize the total operation cost, in which case the MinSum objective should be used; alternatively, the aim may be to minimize the largest service time, in which case the MinMax objective is appropriate.

The collection depots problem was first introduced by Drezner and Wesolowsky [11]. They investigated the MinSum version of the problem, and presented properties of its solutions and an iterative procedure that converges to local minima. The MinSum problem with both Euclidean and rectilinear distances on the plane as well as on a line were also examined.

Properties of solutions to MinMax and MinSum versions of the collection depots location problem on networks were investigated by Berman et al. [6].

Tamir and Halman [25] extensively studied various versions of the MinMax version of the problem, with the additional assumption that the choice of depots available to each customer is restricted to a subset of  $D$ . We will refer to this as the *restricted collection depots problem*. They investigated two variants of one-way versions, in which one of the round trip's legs involving the service center is omitted. They also examined path and tree network versions of the problem. Further results on tree networks have been reported in [5].

Heuristic techniques for solving the MinSum version of the problem with multiple facilities on a network and properties of the solutions are presented in [7].

It is natural to ask how many different values for the depot assignment vector  $I_s$  exist for any placement of the facility in the plane. An obvious upper bound is  $O(p^n)$ , but tighter bounds should exist. Drezner and Wesolowsky [11] left this question open. In this paper (Section 2.3) we show that the bound is  $O(p^2 n^2)$  when the distance metric is Euclidean, and it is tight in the worst case. In addition, the depots assignment vectors can be generated in  $O(p^2 n^2 \log^*(pn))$  time. Thus the collection depots problem can be transformed to  $O(p^2 n^2)$  classical MinMax and MinSum facility location problems. Tamir and Halman [25] gave an  $O(p^2 n^2 \log^3(pn))$  algorithm for the MinMax collection depots problem using the parametric search technique [20] (in Section 3, we describe the algorithm of Tamir and Halman in this new setting). Classical MinSum has been shown to be not exactly solvable using compass and rulers [4], though many practical numerical methods exist. In Section 4, we show how, for the first time, the MinSum collection depots problem can be solved using the classical MinSum algorithm as a subroutine. In Section 5, we show the existence of small coresets for both MinMax and MinSum single facility collection depots problems. A coreset is a subset of input such that we can get a good approximation to the original problem by solving the optimization problem directly on the coreset. The concept of coresets was introduced by Agarwal et al. [2], and a survey of the subject can be found in [3]. The small size coreset is computed by first computing constant factor approximation solutions to the MinMax and MinSum problems. We then describe our  $(1 + \epsilon)$ -factor approximation algorithms for both MinMax and MinSum problems. These algorithms have linear running times for fixed customer weights and constant  $\epsilon$ . In Section 6, we consider location problems involving local barriers [19] and an extension to this problem involving a set of rooms with walls and doors. These problems are examples of the customer one-way model described by Tamir and Halman [25], and in this paper we show how Voronoi diagrams can be used to solve these problems as well.

## 2. General properties

In this section some elegant properties of the collection depots problem are described. A customer  $c$  and depots  $d_1$  and  $d_2$  partition the plane into two regions  $R_1$  and  $R_2$  in such a way that for any point  $q$  in  $R_1$ , the round trip from  $q$  to  $c$  through  $d_1$  is smaller than the round trip from  $q$  to  $c$  through  $d_2$ , and similarly for any point  $q$  in  $R_2$ , the round trip from  $q$  to  $c$  through  $d_2$  is smaller than the round trip from  $q$  to  $c$  through  $d_1$ .

**Definition.** The *round trip distance* for an ordered set of points  $(a_1, \dots, a_k)$  is

$$\text{dist}(a_1, a_2) + \text{dist}(a_2, a_3) + \dots + \text{dist}(a_k, a_1)$$

and is denoted  $\text{rtd}(a_1, \dots, a_k)$ .

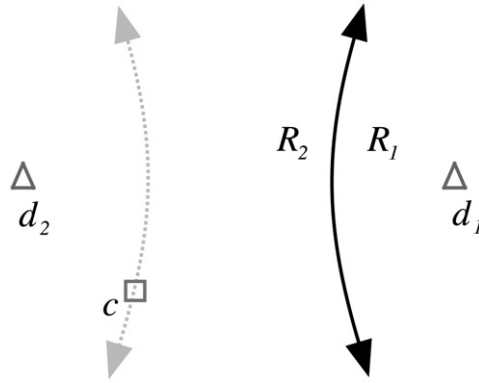


Fig. 1. The partitioning curve of two depots.

Let the curve that partitions the plane into  $R_1$  and  $R_2$  be denoted by  $\eta$ . Note that any point  $q$  on  $\eta$  must satisfy

$$\text{rtd}(q, c, d_1) = \text{rtd}(q, c, d_2) \quad (5)$$

which simplifies to

$$\text{dist}(c, d_1) + \text{dist}(d_1, q) = \text{dist}(c, d_2) + \text{dist}(d_2, q) \quad (6)$$

**Lemma 2.1.** *The partitioning curve  $\eta$  will lie on a (possibly degenerate) hyperbola, and if the hyperbola consists of two separate arms, one arm will contain the customer and the other will contain  $\eta$ .*

**Proof.** Eq. (6) can be rewritten as

$$\text{dist}(d_2, q) - \text{dist}(d_1, q) = \text{dist}(c, d_1) - \text{dist}(c, d_2) \quad (7)$$

Any point  $q \in \eta$  that satisfies (7) must also satisfy

$$|\text{dist}(d_2, q) - \text{dist}(d_1, q)| = |\text{dist}(c, d_1) - \text{dist}(c, d_2)| \quad (8)$$

Since the right-hand side of (8) is a constant, it is the equation of a hyperbola with foci at  $d_1$  and  $d_2$ .

If  $c$  is equidistant to  $d_1$  and  $d_2$ , then the right-hand side of (8) is zero, the hyperbola does not have two separate arms, and we're done. Otherwise, setting  $q$  to  $c$  will satisfy (8), so  $c$  must lie on some arm. Without loss of generality, assume  $\text{dist}(c, d_1) > \text{dist}(c, d_2)$ . Then both sides of (7) are positive, and any point  $q \in \eta$  must be closer to  $d_1$  than  $d_2$ . Since  $c$  is farther from  $d_1$  than  $d_2$ ,  $q \neq c$ ; thus  $\eta$  cannot lie on the same arm as  $c$ .  $\square$

Fig. 1 shows an example of a partitioning curve for two depots. Depots are displayed as triangles, and customers are displayed as squares.

**Lemma 2.2.** *Region  $R_1$  will be empty iff  $d_2$  lies on the line segment between  $c$  and  $d_1$ .*

**Proof.** Let  $L$  be the described line segment. If  $d_2$  lies on  $L$ , then assume some  $q \in R_1$  exists. The smallest round trip between  $c$ ,  $d_1$ , and  $q$  must be strictly less than the smallest round trip between  $c$ ,  $d_2$ , and  $q$ . But  $d_2$  lies on  $L$ , so any smallest round trip involving  $d_1$  must also be a round trip involving  $d_2$ , a contradiction. Thus no  $q$  exists, and  $R_1$  is empty.

If  $R_1$  is empty, then assume  $d_2$  does not lie on  $L$ . Consider  $q$  at  $d_1$ . The shortest round trip between  $c$  and  $q$  has length  $2|L|$ , and by the triangle inequality, any round trip between  $c$ ,  $q$ , and  $d_2$  must have length greater than  $2|L|$ ; thus  $R_1$  must contain  $q$ , a contradiction.  $\square$

**Lemma 2.3.** *If  $d_2$  lies on the line segment between  $c$  and  $d_1$ , and  $c$ ,  $d_1$  and  $d_2$  are distinct, then  $\eta$  is the ray at  $d_1$  pointing away from  $c$ .*

**Proof.** It is easy to show that by the triangle inequality, any point  $q$  satisfying (6) must lie on the line containing the depots and customer. It is also easy to show that only those points on the specified ray satisfy the lemma. Note that the ray is still the arm of a degenerate hyperbola, albeit a degenerate one.  $\square$

Eq. (6) can be rewritten as

$$\text{dist}(d_1, q) + \text{dist}(c, d_1) = \text{dist}(d_2, q) + \text{dist}(c, d_2) \quad (9)$$

If we let  $\alpha_1 = \text{dist}(c, d_1)$  and  $\alpha_2 = \text{dist}(c, d_2)$ , then (9) becomes

$$\text{dist}(d_1, q) + \alpha_1 = \text{dist}(d_2, q) + \alpha_2 \quad (10)$$

Since  $\alpha_1$  and  $\alpha_2$  are both nonnegative, (10) satisfies the equation of edges for the additively weighted Voronoi diagram with sites at the depot locations  $\{d_1, \dots, d_p\}$ , and additive weights equal to the distance of each site from the customer  $c$ . We denote such a Voronoi diagram for a customer  $c$  and depots  $D$  as  $V_{c,D}$ , or as  $V_D$  when  $c$  is clear from the context.

**Definition.** The *Voronoi region for a depot  $d$*  is the set of all points in the  $\mathbb{R}^2$  plane that are at least as close to  $d$  as any other site, according to the round trip distance involving the point, the customer, and the depot. We denote this region as  $U(d)$ . Formally,

$$U(d) = \{q \in \mathbb{R}^2 \mid \forall d' \in D \text{ rtd}(q, c, d) \leq \text{rtd}(q, c, d')\}$$

**Lemma 2.4.** Each  $U(d)$  is unbounded.

**Proof.** Consider the ray from  $d$  along  $\vec{cd}$ . Clearly every point  $q$  on this ray belongs to  $U(d)$ , since the line segment from  $c$  to  $q$  contains  $d$ , and is thus a minimal round trip for a service center located at  $q$ .  $U(d)$  therefore contains this unbounded ray.  $\square$

With general additively weighted Voronoi diagrams, it is possible for some regions to be empty. For example, if the sites consist of two distinct points,  $a$  and  $b$ , and the weights are  $w_a = 0$ ,  $w_b = \text{dist}(a, b) + \epsilon$  for some  $\epsilon > 0$ , then any point  $q$  which is within  $U(b)$  must satisfy

$$\text{dist}(q, b) + \text{dist}(a, b) + \epsilon \leq \text{dist}(q, a)$$

which violates the triangle inequality. We say that  $b$  is *trivial* [18].

**Lemma 2.5.** In the collection depots location problem, no depots are trivial.

**Proof.** This is easy to see; consider when the service facility is located at the depot.  $\square$

**Definition.** A region  $R$  is *star-shaped* if there exists a point  $f$  within  $R$  such that for every point  $q$  in  $R$ , the line segment from  $f$  to  $q$  lies within  $R$ . We also say that  $R$  is *star-shaped with respect to  $f$* .

**Lemma 2.6.** Each  $U(d)$  is star-shaped with respect to  $d$ .

**Proof.**  $U(d)$  is the intersection of some number of unbounded regions bounded by hyperbola arcs (see Fig. 1). Since  $d$  is a focus of each of these hyperbolic arms, each unbounded region is star-shaped with respect to  $d$ ; thus an intersection of some number of them must be as well.  $\square$

We are now ready to establish a bound on the complexity of a Voronoi diagram for a customer.

**Lemma 2.7.** The complexity of  $V_D$  is  $O(|D|)$ .

**Proof.** Lemma 2.6 ensures that each region of  $V_D$  is connected. The proof is thus essentially the same as that given for unweighted Voronoi diagrams in Section 5.3.1 of [21].  $\square$

## 2.1. Ellipse sets

In [25], Tamir and Halman investigated the role of a union of ellipses in the collection depots location problem. In this section, we will further examine this geometric object, and will see how it provides an intuition in the next section for an efficient algorithm for constructing Voronoi diagrams for a customer.

Consider a single customer  $c$  and an upper bound  $r$  on the length of any round trip involving  $c$ , some depot, and a service center. If a particular depot  $d_j$  can participate in such a round trip, then from (1) we have

$$\text{dist}(s, c) + \text{dist}(c, d_j) + \text{dist}(d_j, s) \leq r \quad (11)$$

Note that the weighted travel distance is obtained when the round trip distance is multiplied by the weight of the customer  $c$ . We wish to determine where a service center  $s$  can be located that doesn't violate the bound on  $r$ . Note that for fixed  $c$ ,  $d_j$ , and  $r$ ,  $\text{dist}(c, d_j)$  is a constant, and (11) becomes

$$\text{dist}(s, c) + \text{dist}(d_j, s) \leq t \quad (12)$$

where  $t$  is a constant.

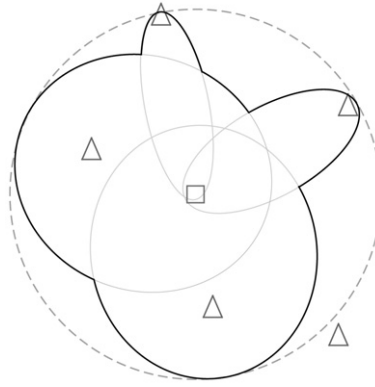


Fig. 2. Ellipse set with bounding circle.

If  $t$  is less than the distance between the customer and depot, (12) has no solution. Otherwise, (12) is the equation of an ellipse with foci at  $c$  and  $d_j$ , with round trip distance  $r = t + \text{dist}(c, d_j)$ ; we will denote this ellipse  $E_{c,d_j,r}$ , or when the customer location and round trip distance are clear from the context,  $E_j$ .

Each depot  $d_j$  is associated with some ellipse  $E_j$ , or cannot participate in any round trips; in this case, we assume  $E_j = \emptyset$ . The set of all possible service center locations for the customer  $c$  that allow a trip length not exceeding  $r$  is then

$$\bigcup_{j=1}^p E_{c,d_j,r}$$

This is a union of a number of ellipses, each with  $c$  as a focus. We will refer to this construct as an *ellipse set for customer  $c$  and round trip length  $r$* , and will denote it  $S_{c,r}$  (or  $S_r$ , if  $c$  is clear from the context).

**Lemma 2.8.** *A bounding circle can be drawn around an ellipse set  $S_{c,r}$  such that every ellipse in the set touches the circle. The radius of this circle is  $r/2$ .*

**Proof.** Consider the ray  $R$  from  $c$  that contains a depot  $d_j$ . If  $d_j = c$ , then  $E_{c,d_j,r}$  is equal to the bounding circle. Otherwise, the farthest point  $q \in R$  from  $c$  that can be part of a round trip with length not exceeding  $r$  is the point at distance  $r/2$  from  $c$ . Every ellipse  $E_j \in S$  will contain exactly one such point  $q_j$ ; thus a circle with radius  $r/2$  centered at  $c$  will touch every ellipse.  $\square$

Note that the diameter of an ellipse set's bounding circle is the upper bound on the round trip distance involving the ellipse set's customer. See Fig. 2 for an example of an ellipse set.

## 2.2. Computing $V_{c,D}$ : The Circle Sweep algorithm

Circle Sweep is an algorithm for producing collection depots Voronoi diagrams, and it is motivated by the relationship between the ellipse set for a customer and its Voronoi diagram.

The first algorithm to generate additively weighted Voronoi diagrams in  $O(n \log n)$  time was presented, though not well developed, by Fortune [14]. While his algorithm is capable of generating  $V_{c,D}$ , we can take advantage of the restricted nature of our additive weights to generate  $V_{c,D}$  in a more intuitive manner while achieving the same time and space complexity.

**Definition.** An *interface point* in the boundary of an ellipse set is a point located at the intersection of two of the boundary's elliptic arcs.

The following properties are easy to prove.

**Lemma 2.9.** *The boundary of ellipse set  $S_{c,r}$  contains exactly those points  $q$  that satisfy*

$$\min_{d \in D} \text{rtd}(c, d, q) = r$$

**Lemma 2.10.** *Every point in the  $\mathbb{R}^2$  plane is contained in the boundary of  $S_{c,r}$ , for exactly one  $r \geq 0$ .*

Voronoi edges contain exactly those points that belong to two or more Voronoi regions, which leads to the following lemmas.

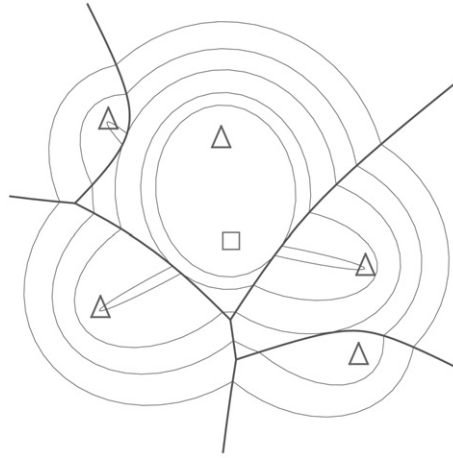


Fig. 3. Voronoi diagram with ellipse sets.

**Lemma 2.11.** Every point in a Voronoi edge lies on an interface point of some ellipse set  $S_{c,r}$ .

**Proof.** By Lemma 2.9, all points with minimal round trip distance  $r$  will lie on the boundary of  $S_{c,r}$ . For a point  $q$  to be on a Voronoi edge, there must be two or more depots admitting an optimal round trip distance  $r$ . Thus there must exist ellipses  $E_1 = E_{c,d_1,r}$  and  $E_2 = E_{c,d_2,r}$  such that  $q$  lies on the boundaries of both  $E_1$  and  $E_2$ , as well as being on the boundary of  $S_{c,r}$ :

$$q \in \partial E_1 \cap \partial E_2 \cap \partial S_{c,r}$$

Clearly  $q$  can only exist where two ellipses intersect on the boundary of  $S_{c,r}$ .  $\square$

**Lemma 2.12.** The interface points of ellipse set  $S_{c,r}$  will trace out the collection depots Voronoi diagram, as  $r$  ranges from  $0 \dots +\infty$ .

**Proof.** Follows from Lemma 2.11.  $\square$

Fig. 3 shows a Voronoi diagram overlaid with ellipse set boundaries for several values of  $r$ .

The Circle Sweep algorithm uses the space sweep technique [15] to generate the vertices and edges of the Voronoi diagram. Instead of using a sweep line or plane, Circle Sweep uses a sweep circle, which represents the radius of an ellipse set's bounding circle. As the radius of the sweep circle increases, the algorithm keeps track of the hyperbolic arcs representing the paths of individual interface points which form the Voronoi edges, and generates Voronoi vertices when two such arcs intersect.

The algorithm uses the standard data structures for a space sweep, an event queue and frontier. The event queue  $Q$  maintains a list of events, sorted in nondecreasing order according to the radius that the sweep circle will have when the event is to be processed. The frontier  $F$  maintains a list of edges, which are portions of the curve partitioning the plane between two depots. An edge separating depot  $d_a$  from  $d_b$  is denoted  $H_{a,b}$ .

The Circle Sweep algorithm, which is reminiscent of Guibas and Stolfi's presentation of Fortune's algorithm for ordinary Voronoi diagrams [15], manipulates three types of event. Each event is a tuple, and the first element of the tuple is the sweep circle radius value the event is sorted by.

- **SITE (radius  $r$ , depot  $d$ ).** At the start of the algorithm, the queue is populated with a SITE event for each depot, where  $r$  is the distance of the depot from the customer. When a SITE event is processed, the existing depot  $d_e$  of the region which contains  $d$  is determined, and the partitioning curve of these two depots is constructed and split into two edges bounding the new region for  $d$ . These edges are added to the frontier, and INTERSECT and WRAP events are predicted for each.
- **INTERSECT (radius  $r$ , point  $p$ , right edge  $H_{a,b}$ , left edge  $H_{b,c}$ ).** Whenever the frontier is modified, either by inserting, deleting, or changing the relative positions of edges, an INTERSECT event is predicted for edges that have become neighbors. When an INTERSECT event is processed, the two edges involved are examined to see if they are still neighbors. It is possible that the frontier has been modified since the prediction was made to the extent that the edges are no longer neighbors; if so, the event is treated as a 'false alarm'. Otherwise, two edges meet at a Voronoi vertex and are replaced by a third edge.
- **WRAP (radius  $r$ , edge  $H$ ).** The frontier maintains a linear list of edges sorted by polar angle, yet the sweep circle represents a closed curve. WRAP events are predicted and processed to move edges whose leading points have crossed from one side of the  $\pm\pi$  ray to the other.

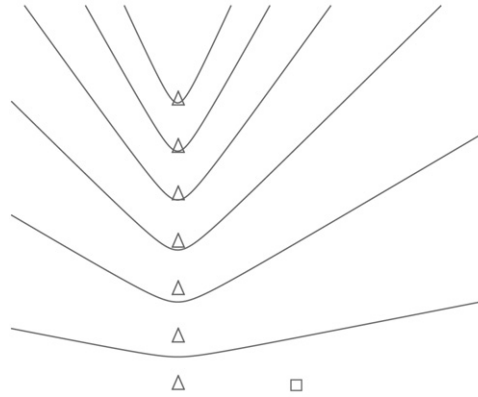


Fig. 4. Voronoi diagram for one customer.

When the event queue is empty, any edges remaining in the frontier will never intersect, and thus are unbounded Voronoi edges. For each of these edges, a point on the hyperbolic arm representing a point at infinity is stored as the final vertex, and the edge is added to the diagram.

**Lemma 2.13.** *The Circle Sweep algorithm constructs  $V_c$  in  $O(p \log p)$  time and  $O(p)$  space.*

**Proof.** Omitted.  $\square$

### 2.3. Feasible assignments

Our approach to solving the collection depots location problem relies on enumerating the possible assignments of depots to customers that can exist for any choice of service center.

**Theorem 2.14.** *At most  $O(p^2 n^2)$  different feasible assignments of depots are possible for any choice of a service center in the Euclidean metric.*

**Proof.** Since  $V_{c,D}$  for each customer has  $O(p)$  complexity, and any edge in one such diagram can intersect every edge in every other diagram at most four times, there are at most  $O((pn)^2)$  regions in the merged diagrams. For any point in a particular region, the depots assignments for the customers of  $C$  remains the same.  $\square$

**Lemma 2.15.** *All different feasible assignments of depots can be computed in  $O(p^2 n^2 \log^* pn)$  time.*

**Proof.** By Lemma 2.13, each  $V_{c,D}$  can be constructed in  $O(p \log p)$  time. Using the results in [12] one can see that the complexity of the zone of a hyperbolic arc  $\delta$  in an arrangement of  $pn$  hyperbolic arcs, each pair of which intersects in at most four points, is  $\lambda_6(pn)$ . Here the function  $\lambda_6(pn)$  is defined as the maximum length of a Davenport–Schinzel sequence of order six on  $pn$  symbols, and it is almost linear in  $pn$ . As a matter of fact,  $\lambda_{s+2}(pn) \in O(pn \log^* pn)$  for any constant  $s$  [12]. Therefore, the  $n$  Voronoi diagrams  $V_{c,D}$  can be incrementally merged in  $O(p^2 n^2 \log^* pn)$  time.  $\square$

It is possible to construct an example to show that the above bound is tight. First, we start with a vertical line of depots. A single customer  $c$  to the right of the lowest depot produces  $V_{c,D}$  of Fig. 4.

Adding a second customer to the right of the highest depot produces Fig. 5. The idea can be extended by adding additional customers to the right of the existing customers (Fig. 6). In this way, we can generate an example whose number of feasible assignments of depots is  $\Theta(p^2 n^2)$ .

### 3. MinMax problem

Tamir and Halman [25] presented an  $O(p^2 n^2 \log^3(pn))$  algorithm for the MinMax restricted collection depots problem. The algorithm is based on the parametric approach of Megiddo [20] which requires an efficient parallel implementation for the following decision problem (called covering problem): Determine whether there exists a facility location such that the maximum round trip cost of the customers of  $C$  is at most  $r$ .

**Definition.** The *weighted round-trip distance*  $r$  for a customer  $c$  is  $r/w_c$ , and is denoted  $r_c$ .

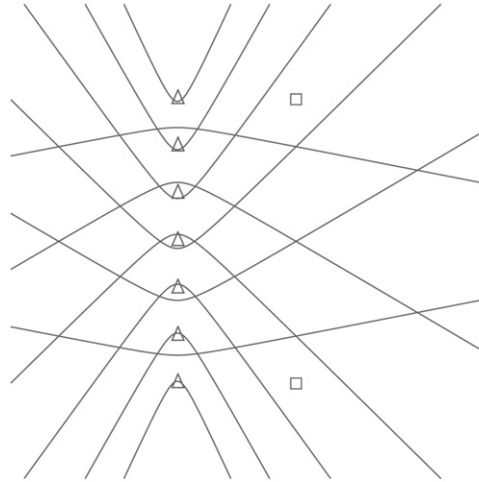


Fig. 5. Voronoi diagrams for two customers.

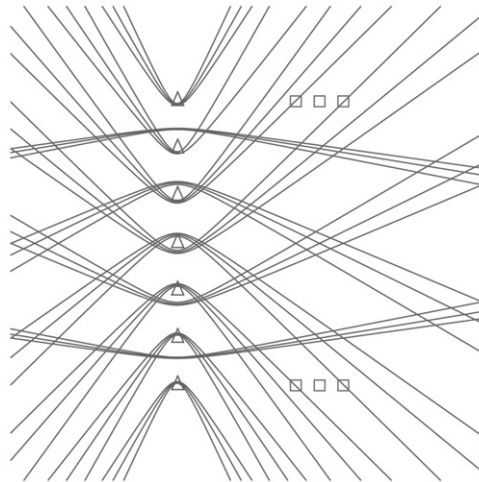


Fig. 6. Worst case situation.

Recall that  $E_{c,d_j,r_c}$  is the ellipse consisting of those service center locations that can participate with customer  $c$  and depot  $d_j$  in a round trip of length not exceeding  $r_c$ . The union of the  $p$  depots gives us the ellipse set  $S_{c,r_c}$ . For the covering problem, we ask the question: Is  $\bigcap_{c \in C} S_{c,r_c}$  empty? It was argued in [25] that the boundary of  $S_{c,r_c}$  can have  $O(p2^{\alpha(p)})$  vertices and elliptical arcs where  $\alpha(p)$  is the functional inverse of the Ackermann's function. We can, in fact, show that its complexity is  $O(p)$ .

**Theorem 3.1.** *The complexity of an ellipse set of  $p$  depots is  $O(p)$ .*

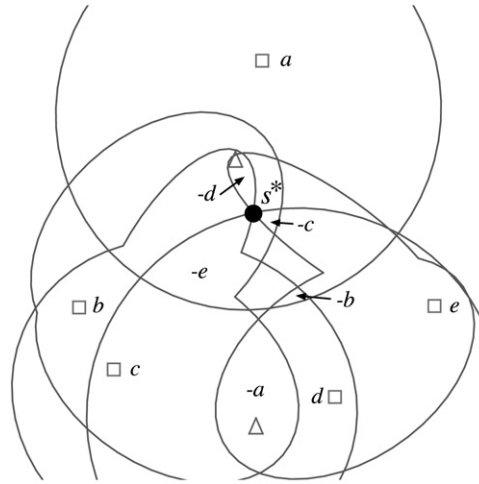
**Proof.** By Lemma 2.7,  $V_{c,D}$  has  $O(p)$  edges. If we start the Circle Sweep algorithm, and interrupt it when the sweep circle has radius  $r/2$ , the frontier  $F$  will contain a list of edges from which the set of ellipse segments comprising  $S_{c,r}$  can be easily extracted. Since  $V_{c,D}$  has  $O(p)$  edges,  $F$  has  $O(p)$  edges; thus there are  $O(p)$  segments in  $S_{c,r}$ .  $\square$

**Theorem 3.2.** *An ellipse set of  $p$  depots can be constructed in  $O(p \log p)$  time.*

**Proof.** By Lemma 2.13, the frontier  $F$  of the proof of Lemma 3.1 can be constructed in  $O(p \log p)$  time. Extracting the ellipse segments by iterating through  $F$  can be done in  $O(p \log p)$  time as well.  $\square$

Whether  $\bigcap_{c \in C} S_{c,r_c}$  is nonempty can be tested in  $O(p^2 n^2 \log(pn))$  time using the standard plane-sweep algorithm. The optimal value of the MinMax collection depots problem is the smallest  $r$  of the covering problem for which  $\bigcap_{c \in C} S_{c,r_c}$  is



Fig. 7.  $\gamma_{C,D} = 5$ .

nonempty. For this the parametric approach of Megiddo [20] is applied with the parallel implementation of the algorithm described in Agarwal et al. [1] (see also Sharir and Agarwal [23]) is used to test whether  $\bigcap_{c \in C} S_{c,r_c}$  is nonempty.

**Theorem 3.3.** (See [25].) *The optimal solution to the MinMax collection depots problem can be computed in  $O(p^2 n^2 \log^3(pn))$  time.*

It is an open problem of whether an intersection of  $n$  ellipse sets of  $p$  depots can have a complexity of  $\Theta(p^2 n^2)$ . Tamir and Halman [24] have achieved this complexity for the restricted collection depots problem. Their technique does not extend to the unrestricted version of the problem, and we conjecture that this complexity is not possible for the unrestricted case.

An interesting question to ask is how many customers are required to determine the optimal location to the MinMax depots problem, for a given set of customers and depots (in the classical planar MinMax problem, this number is at most 3). Let  $\gamma_{C,D}$  represent this number. We conjecture that  $\gamma_{C,D}$  is bounded by some small integer.

It is easy to construct examples where  $\gamma_{C,D} = 4$ . Fig. 7 is an example where  $\gamma_{C,D} = 5$ . The customers are labelled  $a \dots e$ , and the optimal service center location is labelled  $s^*$ . There exist nonempty regions of intersection between every four of the five customers' ellipse sets; these are labelled  $-q$ , where  $q$  is the excluded customer. If customer  $c_q$  is removed from  $C$ ,  $s^*$  will move to the interior of  $-q$ , and the value of Eq. (1) will decrease.

#### 4. MinSum problem

It was observed in [11] that Eq. (4) can be rewritten as follows:

$$G(s) = \sum_{i=1}^n w_i \cdot \text{dist}(s, c_i) + \sum_{i=1}^n w_i \cdot \text{dist}(c_i, I_s[i]) + \sum_{i=1}^n w_i \cdot \text{dist}(I_s[i], s) = G_1(s) + G_2(s) + G_3(s)$$

For a given assignment vector  $I_s$ ,  $G_2(s)$  is constant, so minimizing  $G(s)$  is the same as minimizing  $G_1(s) + G_3(s)$  which is the classical MinSum problem of  $2n$  points (note that a depot may appear more than once in the list). By Lemma 2.15, all feasible vectors  $I_s$  can be determined in  $O(p^2 n^2 \log^*(pn))$  time; therefore

**Theorem 4.1.** *The MinSum collection depots problem can be solved in  $O(p^2 n^2 (T(n) + \log^*(pn)))$  time, where  $T(n)$  is the time it takes to solve the classical MinSum problem of  $O(n)$  points.*

As observed by [4], an exact solution to the classical MinSum problem in the Euclidean plane is not possible. However, it is straightforward to calculate an approximation to this problem using an iterative approach in which each step produces a more accurate approximation. One common approach of this type, called Weiszfeld's algorithm [26], is a form of iteratively re-weighted least squares. Other algorithms are mentioned in [9,27].

Bose et al. [8] have proposed deterministic  $O(n \log n)$  time and randomized  $O(n)$  time  $(1 + \epsilon)$ -approximation algorithms for the classical MinSum problem where the customer weights and  $\epsilon$  are constants. Har-Peled and Mazumdar [17] proposed a randomized linear time algorithm to compute a coresot of the input customer set such that the solution using the coresot realizes a  $(1 + \epsilon)$ -approximation solution to the original problem. The size of the coresot constructed in [17] is a function of  $\epsilon$  and the weights associated with the customers. Note that when the customers are unweighted, it is assumed that  $w_i = 1$  for all  $1 \leq i \leq n$ . A further improvement to the coresot construction was proposed in [16]. Thus for fixed customer weights

and  $\epsilon$ , we can compute a  $(1 + \epsilon)$ -approximate solution to the MinSum collection depots problem in  $O(p^2 n^3)$  expected time or in  $O(p^2 n^3 \log n)$  worst case time.

In the next section we will show how we can compute a  $(1 + \epsilon)$ -approximate solution in linear time for fixed  $\epsilon$  and fixed customer weights.

## 5. Approximation algorithms

The exact algorithms proposed for the MinMax and MinSum problems are expensive for large  $p$  and  $n$ . We first present here simple constant factor approximation algorithms for both the MinMax and MinSum collection depots problems, and then employ these solutions to compute a small size  $\epsilon$ -coreset for the MinMax and MinSum collection depots problem.

### 5.1. The MinSum depots problem

**Definition.** The *centroid* of a set of  $n$  points  $p_i \in \mathbb{R}^2$  with weights  $w_i$  is

$$\frac{\sum_{i=1}^n (w_i \cdot p_i)}{\sum_{i=1}^n w_i}$$

**Lemma 5.1.** *Using the centroid of the customers as the service center location is a 3-approximation to the MinSum depots problem.*

**Proof.** Let  $t^*$  be the optimal MinSum service center location, and  $\tilde{t}$  the centroid of the customers. As shown in [22], the centroid is a 2-approximation solution for the classical MinSum problem. Therefore

$$\begin{aligned} \sum_{i=1}^n (w_i \cdot \text{rtd}(\tilde{t}, c_i, I_{\tilde{t}}[i])) &\leq \sum_{i=1}^n (w_i \cdot \text{rtd}(\tilde{t}, c_i, I_{t^*}[i])) \leq \sum_{i=1}^n (w_i \cdot \text{rtd}(c_i, \tilde{t}, c_i, I_{t^*}[i], t^*)) \\ &= \sum_{i=1}^n (w_i \cdot \text{rtd}(c_i, \tilde{t})) + G(t^*) \leq 3 \cdot G(t^*) \quad \square \end{aligned}$$

We now use this 3-approximate solution to the MinSum depots problem to develop a  $(1 + \epsilon)$ -approximation solution to the MinSum depots problem, based on the concept of coresets. A coreset of an optimization problem is a small representative set of input, such that one can get a good approximation to the original input by solving the optimization problem directly on the coreset. To do so, we will use the same construction as that employed by Har-Peled and Mazumdar [17].

Let  $W = \sum_{i=1}^n w_i$ ,  $\tilde{W} = W/w_{\min}$ , and  $R^* = \frac{G(t^*)}{W}$ . Let  $\tilde{R} = \frac{G(\tilde{t})}{3\tilde{W}}$ , where  $\tilde{t}$  is the centroid of the customer set. Observe that  $\tilde{R}$  is a lower bound of  $R^*$ . Note that since the weighted round trip cost between  $\tilde{t}$  and any one customer cannot exceed  $3\tilde{R}\tilde{W}$ , no customer can be farther than  $\frac{3}{2}\tilde{R}\tilde{W}$  from  $\tilde{t}$ . Similar reasoning shows that with the service center at  $t^*$ , the distance between a customer and its depot in a shortest round trip cannot exceed  $\frac{3}{2}\tilde{R}\tilde{W}$ ; thus we ignore any depot farther than  $3\tilde{R}\tilde{W}$  from  $\tilde{t}$ . Clearly such depots can be identified and removed in linear time. We now construct an exponential grid around  $\tilde{t}$  similar to the one used in [17].

Let  $Q_j$  be the axis-parallel square with side length  $\tilde{R} \cdot 2^j$  centered at  $\tilde{t}$ , with  $j = 0, 1, \dots, M$  where  $M = \lceil \log 6\tilde{W} \rceil$ . Let  $V_0 = Q_0$ , and  $V_j = Q_j \setminus Q_{j-1}$ . We partition  $V_j$  into a grid with side length  $r_j = \frac{\epsilon \tilde{R} 2^j}{2\beta}$  for some  $\beta$  to be determined later.

For each grid cell in  $V_j$  we determine the customers inside the cell. We select an arbitrary point inside the cell as the representative customer for all customers within the cell, and assign it a weight equal to the sum of the weights of the customers within the cell. We denote the representative customer for any customer  $c$  by  $c'$ . We also select an arbitrary point inside each cell as the representative depot  $d'$  for all depots  $d$  within the cell. Let  $C'$  and  $D'$  denote the mapped locations of the customers  $C$  and depots  $D$  respectively.

Since each grid cell in  $V_j$  has sides of length  $\frac{\epsilon \tilde{R} 2^j}{2\beta}$ , its diagonal distance is bounded above by  $\frac{\epsilon \tilde{R} 2^j}{\beta}$ . Hence, for a point  $q$  in a grid cell of  $V_j$  (in which  $\text{dist}(q, \tilde{t}) \geq \tilde{R} 2^{j-2}$ ), where  $j \geq 1$ ,

$$\text{dist}(q, q') \leq \frac{\epsilon \tilde{R} 2^j}{\beta} = \frac{4\epsilon \tilde{R} 2^{j-2}}{\beta} \leq \frac{4\epsilon}{\beta} \text{dist}(q, \tilde{t})$$

**Lemma 5.2.**  $|C'| + |D'| = O(\frac{\log \tilde{W}}{\epsilon^2})$ .

**Proof.** Let  $|V_j|$  be the number of grid cells in  $V_j$ . Then

$$|V_j| \leq \left\lceil \frac{\tilde{R} 2^j}{r_j} \right\rceil^2 \in O\left(\frac{1}{\epsilon^2}\right)$$

Therefore, the total number of cells in  $V_0, V_1, \dots, V_M$  is  $O(\frac{\log W}{\epsilon^2})$ .  $\square$

Note that  $C'$  and  $D'$  can be computed in linear time once  $\tilde{t}$  is known.

**Lemma 5.3.** *The coresets  $C'$  and  $D'$  constitute an  $\epsilon$ -coreset for  $C$  and  $D$ .*

**Proof.** Let  $q$  be an arbitrary point in the plane. The error incurred for using  $C'$  and  $D'$  instead of  $C$  and  $D$  when the service center is located at  $q$  is

$$E \leq \sum_{i=1}^n (w_i \cdot |\text{rtd}(q, c'_i, I'_t[i]) - \text{rtd}(q, c_i, I_t[i])|)$$

where  $I'_t[i]$  is the representative depot corresponding to  $I_t[i]$ . By the triangle inequality,

$$\begin{aligned} \text{rtd}(q, c_i, I_t[i]) &= \text{dist}(q, c_i) + \text{dist}(c_i, I_t[i]) + \text{dist}(I_t[i], q) \\ &\geq \text{dist}(q, c'_i) - \text{dist}(c_i, c'_i) \\ &\quad + \text{dist}(c'_i, I'_t[i]) - \text{dist}(c_i, c'_i) - \text{dist}(I_t[i], I'_t[i]) \\ &\quad + \text{dist}(I'_t[i], q) - \text{dist}(I_t[i], I'_t[i]) \\ &= \text{rtd}(q, c'_i, I'_t[i]) - 2 \text{dist}(c_i, c'_i) - 2 \text{dist}(I_t[i], I'_t[i]) \end{aligned}$$

and by symmetry we can show

$$\text{rtd}(q, c'_i, I'_t[i]) \geq \text{rtd}(q, c_i, I_t[i]) - 2 \text{dist}(c_i, c'_i) - 2 \text{dist}(I_t[i], I'_t[i])$$

Thus we have

$$E \leq 2 \sum_{i=1}^n w_i (\text{dist}(c_i, c'_i) + \text{dist}(I_t[i], I'_t[i]))$$

We can write

$$\sum_{i=1}^n w_i \text{dist}(c_i, c'_i) = \sum_{c_i \in Q_0} w_i \text{dist}(c_i, c'_i) + \sum_{c_i \notin Q_0} w_i \text{dist}(c_i, c'_i) \quad (13)$$

Clearly

$$\sum_{c_i \in Q_0} w_i \text{dist}(c_i, c'_i) \leq \sum_{i=1}^n w_i \frac{\epsilon \tilde{R} 2^0}{\beta} \leq \frac{\epsilon}{\beta} \tilde{R} \cdot W$$

Since  $\text{dist}(c_i, c'_i) \leq \frac{4\epsilon}{\beta} \text{dist}(c_i, \tilde{t})$  for  $c_i \in V_j$ ,  $j \geq 1$ , we have

$$\sum_{c_i \notin Q_0} w_i \text{dist}(c_i, c'_i) \leq \sum_{i=1}^n w_i \frac{4\epsilon}{\beta} \text{dist}(c_i, \tilde{t}) = \frac{4\epsilon}{\beta} \sum_{i=1}^n w_i \text{dist}(c_i, \tilde{t})$$

Eq. (13) becomes

$$\sum_{i=1}^n w_i \text{dist}(c_i, c'_i) \leq \frac{\epsilon}{\beta} \tilde{R} \cdot W + \frac{4\epsilon}{\beta} \sum_{i=1}^n w_i \text{dist}(c_i, \tilde{t})$$

Similarly, we can write

$$\sum_{i=1}^n w_i \text{dist}(I_t[i], I'_t[i]) \leq \frac{\epsilon}{\beta} \tilde{R} \cdot W + \frac{4\epsilon}{\beta} \sum_{i=1}^n w_i \text{dist}(I_t[i], \tilde{t})$$

therefore

$$\begin{aligned} E &\leq \frac{4\epsilon}{\beta} \tilde{R} \cdot W + \frac{8\epsilon}{\beta} \sum_{i=1}^n w_i (\text{dist}(c_i, \tilde{t}) + \text{dist}(I_t[i], \tilde{t})) \\ &\leq \frac{4\epsilon}{\beta} \tilde{R} \cdot W + \frac{8\epsilon}{\beta} 3 \cdot \tilde{R} \cdot W \\ &= \frac{28}{\beta} \cdot \epsilon \cdot \tilde{R} \cdot W \\ &\leq \frac{28}{\beta} \cdot \epsilon \cdot R^* \cdot W \end{aligned}$$

Thus, for  $\beta \geq 28$ ,  $E \leq \epsilon \cdot G(t^*)$ .  $\square$

**Theorem 5.4.** *The  $\epsilon$ -coresets  $C'$  and  $D'$  realize a  $(1 + 3\epsilon)$ -approximate solution to the MinSum planar collection depots location problem, and can be computed in linear time for fixed  $\epsilon \leq 1$  and constant customer weights.*

**Proof.** Our  $\epsilon$ -coresets  $C'$ ,  $D'$  have  $k = O(\frac{\log \tilde{W}}{\epsilon^2})$  elements. By Lemma 5.3, these coresets can be generated in  $O(n + p)$  time, for which the optimal solution  $\tilde{t}^*$  is a  $(1 + \epsilon)$ -factor approximate solution for  $t^*$ . By Lemma 2.15, these coresets yield at most  $O(k^4)$  feasible assignments of depots to customers, and these can be generated in  $O(k^4 \log^* k)$  time. Each feasible assignment represents a classical MinSum problem, for which a  $(1 + \epsilon)$ -factor approximate solution,  $\tilde{t}$ , can be produced in  $O(k)$  expected time [16,17] or  $O(k \log k)$  deterministic time [8]. Therefore

$$\begin{aligned} \sum_{i=1}^n (w_i \cdot \text{rtd}(\tilde{t}, c_i, I_{\tilde{t}}[i])) &\leq (1 + \epsilon) \cdot \sum_{i=1}^n (w_i \cdot \text{rtd}(\tilde{t}^*, c_i, I_{\tilde{t}^*}[i])) \\ &\leq (1 + \epsilon)^2 \cdot \sum_{i=1}^n (w_i \cdot \text{rtd}(t^*, c_i, I_{t^*}[i])) \\ &\leq (1 + 3\epsilon) \cdot G(t^*) \end{aligned}$$

for  $\epsilon \leq 1$ . Thus a  $(1 + 3\epsilon)$ -factor approximate solution can be determined in  $O(n + p)$  time.  $\square$

## 5.2. The MinMax depots problem

**Lemma 5.5.** *Using the largest-weighted customer as the service center location is a 2-approximation to the MinMax depots problem.*

**Proof.** Let  $s^*$  be the optimal MinMax service center location, and  $C_m$  the customer with the largest weight. Then

$$\begin{aligned} w_i \cdot \text{rtd}(C_m, c_i, I_{C_m}[i]) &\leq w_i \cdot \text{rtd}(C_m, c_i, I_{s^*}[i]) \\ &\leq w_i \cdot \text{rtd}(s^*, C_m, s^*, c_i, I_{s^*}[i]) \\ &= w_i \cdot \text{rtd}(s^*, C_m) + w_i \cdot \text{rtd}(s^*, c_i, I_{s^*}[i]) \\ &\leq 2 \cdot F(s^*) \quad \square \end{aligned}$$

We can use this approximation to develop a  $(1 + \epsilon)$ -factor approximation for the MinMax depots problem, just as we did for the MinSum depots problem.

Let  $w_{\max} = \max_{i=1, \dots, n} w_i$ ,  $\tilde{W} = w_{\max}/w_{\min}$ , and  $R^* = \frac{F(s^*)}{w_{\max}}$ . Let  $\tilde{R} = \frac{F(\tilde{s})}{2w_{\max}}$ , where  $\tilde{s}$  is the 2-approximation given above. Observe that  $\tilde{R}$  is a lower bound of  $R^*$ . Note that no customers can be farther than  $\tilde{R}\tilde{W}$  from  $\tilde{s}$ , and that we can ignore depots farther than  $2\tilde{R}\tilde{W}$  from  $\tilde{s}$ . We now construct an exponential grid around  $\tilde{s}$ , just as we did in the previous section, but with  $M = \lceil \log 4\tilde{W} \rceil$ . Each grid section  $V_j$  has side length  $r_j = \frac{\epsilon \tilde{R} 2^j}{2\beta}$  for some  $\beta$  to be determined later. We construct  $C'$  and  $D'$ , sets of representative customers and depots.

The proof of the following lemma is analogous to that of Lemma 5.2.

**Lemma 5.6.**  $|C'| + |D'| = O(\frac{\log \tilde{W}}{\epsilon^2})$ .

**Lemma 5.7.** *The coresets  $C'$  and  $D'$  constitute an  $\epsilon$ -coreset for  $C$  and  $D$ .*

**Proof.** Let  $q$  be an arbitrary point in the plane. The error incurred for using  $C'$  and  $D'$  instead of  $C$  and  $D$  when the service center is located at  $q$  is

$$E = \max_{i=1, \dots, n} (w_i \cdot |\text{rtd}(q, c'_i, I'_{\tilde{s}}[i]) - \text{rtd}(q, c_i, I_{\tilde{s}}[i])|)$$

where  $I'_{\tilde{s}}[i]$  is the representative depot corresponding to  $I_{\tilde{s}}[i]$ . By following a procedure similar to that of Lemma 5.3, we can ultimately show that  $E \leq \epsilon \cdot F(s^*)$  for a choice of  $\beta \geq 20$ .  $\square$

**Theorem 5.8.** *The  $\epsilon$ -coresets  $C'$  and  $D'$  realize a  $(1 + \epsilon)$ -approximate solution to the MinMax planar collection depots location problem, and can be computed in linear time for fixed  $\epsilon$  and constant customer weights.*

**Proof.** Our  $\epsilon$ -coresets  $C'$ ,  $D'$  have  $k = O(\frac{\log \tilde{W}}{\epsilon^2})$  elements. By Lemma 5.7, these coresets can be generated in  $O(n + p)$  time. The optimal solution to the MinMax depots problem for these coreset customers and depots can be calculated in  $O(k^4 \log^3 k)$  time [25]. Thus a  $(1 + \epsilon)$ -factor approximate solution can be determined in  $O(n + p)$  time.  $\square$

## 6. Additional applications

We will show how the technique of merging Voronoi diagrams to generate a list of feasible depot assignments can be applied to problems involving line barriers. These involve the customer one-way collection depots problems described by Tamir and Halman [25].

### 6.1. The MinSum barrier problem

Klamroth [19] investigated the classical MinSum problem in the presence of a single line barrier, which can represent a river, highway, or other border.

Let  $\{c_1, \dots, c_n\}$  represent existing facilities (which we will refer to as customers) in the  $\mathbb{R}^2$  plane, each associated with a positive weight  $w_i$ . Without loss of generality, we assume the barrier is a horizontal line with a set  $D = \{d_1, \dots, d_p\}$  of points on the line representing passages across the barrier.

The objective function to be minimized is

$$G(s) = \sum_{i=1}^n w_i \cdot \text{dist}'(s, c_i) \quad (14)$$

where the distance function  $\text{dist}'(a, b)$  is defined as the shortest path from  $a$  to  $b$  that doesn't cross the barrier except possibly at one of the passages. Finding  $s$  which minimizes (14) is the MinSum barrier problem.

Klamroth points out that the optimal  $s$  exists on one of the two sides of the barrier, so the solution to the problem can be viewed as the best of the solutions of the subproblems to either side. To solve a subproblem, partition the customers into two sets  $C^1$  and  $C^2$  according to which side of the barrier they lie on. Then the optimal  $s$  will be that which minimizes one of

$$G^{(1)}(s) = \sum_{c_i \in C^1} w_i \cdot \text{dist}(s, c_i) + \sum_{c_i \in C^2} w_i \cdot \left\{ \min_{j=1, \dots, p} \{ \text{dist}(s, d_j) + \text{dist}(d_j, c_i) \} \right\} \quad (15)$$

$$G^{(2)}(s) = \sum_{c_i \in C^2} w_i \cdot \text{dist}(s, c_i) + \sum_{c_i \in C^1} w_i \cdot \left\{ \min_{j=1, \dots, p} \{ \text{dist}(s, d_j) + \text{dist}(d_j, c_i) \} \right\} \quad (16)$$

Once the assignments of passages to those customers lying on the opposite side of the barrier from  $s$  are made, the final distance terms in (15) and (16) become constants, and the problem reduces to the classical MinSum problem.

Klamroth presented an algorithm which reduces the MinSum barrier problem with  $p$  passages to  $O(\binom{n+p-1}{p-1})$  classical MinSum subproblems. This is an improvement over a simple enumeration of all possible selection of passages, which produces  $O(p^{2n})$  subproblems, yet is still exponential in  $p$ .

To improve these results, we construct, for each customer  $c$ , a Voronoi diagram associating each point  $s$  in the halfplane opposite the customer with the passage  $d_i$  which minimizes  $\text{dist}'(c, s)$ . We restrict the diagram to the halfplane on the opposite side of the barrier from  $c$ , since these are the only points for which passage assignments are necessary.

The partitioning curve between regions for passages  $d_1$  and  $d_2$  are the points  $q$  which satisfy

$$\text{dist}(c, d_1) + \text{dist}(d_1, q) = \text{dist}(c, d_2) + \text{dist}(d_2, q)$$

which is identical to (6). Thus the Voronoi diagram we seek is exactly that portion of  $V_{c,D}$  which lies on the appropriate side of the barrier.

A Voronoi diagram for  $p$  barriers will consist of  $p - 1$  nonintersecting hyperbolic arcs, as Fig. 8 shows.

**Theorem 6.1.** *The MinSum barrier problem with  $n$  customers and  $p$  passages can be solved in  $O(p^2 n^2 (T(n) + \log^*(pn)))$  time, where  $T(n)$  is the time it takes to solve the classical MinSum problem of  $O(n)$  points.*

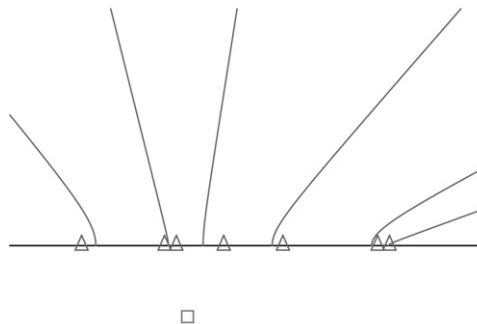


Fig. 8. Voronoi diagram for customer, barrier problem.

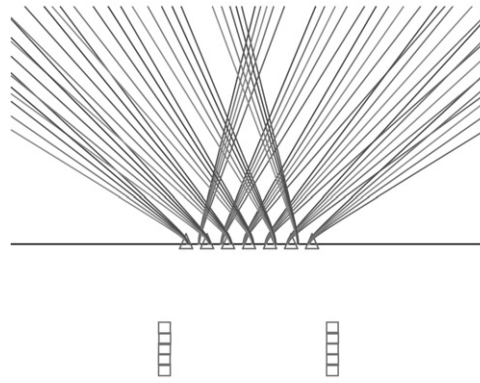


Fig. 9. Worst case situation, barrier problem.

**Proof.** Since the Voronoi diagram for the barrier problem is a subset of  $V_{C,D}$ , we can directly apply Theorems 2.14 and 2.15 to show that at most  $O(p^2n^2)$  different feasible assignments of passages to customers are possible. Each such set of assignments reduces the line barrier problem to the classical MinSum problem.  $\square$

The  $O(p^2n^2)$  bound on the number of feasible assignments in the proof of Theorem 6.1 is tight. Using an approach similar to that employed in Section 2.3, we can construct the example of Fig. 9.

### 6.2. The MinMax barrier problem

If we replace Eq. (14) with the objective function

$$F(s) = \max_{i=1,\dots,n} w_i \cdot \text{dist}'(s, c_i) \quad (17)$$

we get the MinMax barrier problem.

As with the MinSum barrier problem, the approach taken to solve the MinMax barrier problem is to search on both sides of the barrier and choose the best of the two solutions.

To find  $s$  on a particular side of the barrier, the problem is very similar to the MinMax depots problem of Section 3. As observed in [25], instead of ellipse sets, here we are concerned with *circle sets*: a union of circles for each customer. If the customer is on the same side of the barrier as  $s$ , this union will consist of a single circle centered at the customer; otherwise, it will consist of a set of at most  $p$  circles centered at each passage. Each circle in a set has a radius that is additively weighted by the (negative) distance of its center from the customer, and is multiplicatively weighted by the inverse of the customer's weight. In both cases, the circles are cropped to exclude the halfplane which doesn't contain  $s$ .

A circle set represents exactly those points in the halfplane of  $s$  that satisfy  $\text{dist}'(q, c_i) \leq r/w_i$ , for customer  $c_i$  (with weight  $w_i$ ) and maximum weighted distance  $r$ .

**Theorem 6.2.** (See [25].) *The MinMax barrier problem with  $n$  customers and  $p$  passages can be solved in  $O(p^2n \text{ polylog}(pn))$  time.*

### 6.3. The room problem

In this section, we investigate an extension of the barrier problem. Consider  $n$  customers inside a polygon that has been partitioned into  $m$  smaller polygons by a set of linear barriers; see Fig. 10. In this context, we call the smaller polygons *rooms*, the barriers *walls*, and the  $p$  passages *doors*. We assume each room is convex. (The proposed algorithm can also handle nonconvex rooms; the details are tedious, but straightforward.)

We wish to find the location within a particular room that minimizes

$$G(s) = \sum_{i=1}^n w_i \cdot \text{dist}'(s, c_i) \quad (18)$$

which is the same as Eq. (14), but with the distance  $\text{dist}'(a, b)$  representing the shortest path from  $a$  to  $b$  that doesn't cross any walls at points other than doors. We will call the task of finding a location  $s$  within a particular room that minimizes Eq. (18) the *room problem*.

We assume that there exists a network  $N$  with  $p$  nodes, with edges between two nodes if the corresponding doors share a room. The convexity of the rooms ensures that the distance represented by each edge is simply the Euclidean distance between the nodes. In the worst case, the network  $N$  has  $O(p^2)$  edges.

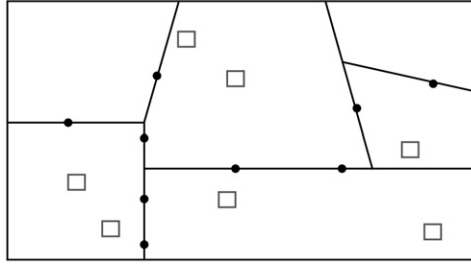


Fig. 10. Polygon partitioned by barriers into convex rooms, with customers and doors.

**Theorem 6.3.** *The room problem with  $n$  customers and  $p$  doors can be solved in  $O(p^2n^2(T(n) + \log^*(pn)))$  time, where  $T(n)$  is the time it takes to solve the classical MinSum problem of  $O(n)$  points.*

**Proof.** For every customer  $c_i$ , we determine the single-source shortest paths from  $c_i$  to every door. This can be done as follows. We first augment the network  $N$  by adding the vertex  $c_i$  and the edges from  $c_i$  in a room to all the doors of the room. We then solve the single source shortest path problem in the augmented network with  $c_i$  as the source node. Dijkstra's algorithm [10] can be applied to solve the problem in  $O(p^2)$  time.

We repeat the process for each customer in  $C$ . Therefore, in  $O(n \cdot p^2)$  time, we can compute the distance of all the customers to all the doors of the rooms. The storage space requirement is  $O(p^2)$ .

Let  $h$  be the room containing  $s$ . As in the previous section, we partition the customers  $C$  into two sets:  $C^1$ , the customers in room  $h$ , and  $C^2$ , those not in room  $h$ . We also construct, for each room  $h$ , the set of doors that exist in  $h$ , and denote this set  $D^h$  (with  $p^{(h)} = |D^h|$ ). Eq. (18) becomes

$$G(s) = \sum_{c_i \in C^1} w_i \cdot \text{dist}(s, c_i) + \sum_{c_i \in C^2} w_i \cdot \min_{j=1, \dots, p^{(h)}} (\text{dist}(s, d_j^h) + \text{dist}'(d_j^h, c_i)) \quad (19)$$

Observe that for fixed room  $h$ ,  $\text{dist}'(d_j^h, c)$  is a constant precomputed by the single source shortest path algorithm, and can be used as an additive weight for customer  $c$  when we construct a Voronoi diagram with the doors from room  $h$  as sites.

We can construct this diagram for a customer and room in  $O(p \log p)$  time, once the additive weights have been determined. We then proceed as in the previous section, merging the  $n$  diagrams together to get  $O(p^2n^2)$  sets of feasible assignments of doors to customers. Each of these sets reduces the problem to the classical MinSum problem of  $n$  points.  $\square$

Note that we can find the optimal  $s$  over all rooms by choosing the best solution from  $m$  room problems.

The MinMax version of the room problem is very similar to the MinMax barrier problem, and can be solved with the same approach and with the same running time given in Theorem 6.2.

## 7. Conclusion

In this paper, we examined both MinMax and MinSum variants of the planar Euclidean collection depots location problem. We have shown how Voronoi diagrams can be used to enumerate candidate solutions to these problems, and how their use yields an improved algorithm for solving a type of MinSum problem with line barriers and its generalization. We can draw the following conclusions:

- We have solved an open problem posed by [11] by proving that at most  $O(p^2n^2)$  different feasible assignments of depots are possible for any choice of service center in the Euclidean metric, and this bound is tight. In addition, these assignments can be generated in time  $O(p^2n^2 \log^*(pn))$ .
- For the first time, we have solved the MinSum collection depots problem by showing that it can be reduced in  $O(p^2n^2 \log^*(pn))$  time to  $O(p^2n^2)$  classical MinSum problems of  $O(n)$  points.
- The MinSum barrier and room problems can be reduced to  $O(p^2n^2)$  classical MinSum problems of  $O(n)$  points. This improves the solution proposed by Klamroth [19] for the barrier problem.
- We have presented linear time  $(1 + \epsilon)$ -approximation algorithms for both the MinMax and MinSum collection depots problems for fixed customer weights and  $\epsilon$ .

Areas for future research include:

- It is an open problem of whether the intersection of customer ellipse sets has complexity of  $o(p^2n^2)$  for the unrestricted collection depots problem.
- It is an open problem of whether a constant bound exists on the number of customers that determine the solution to the MinMax depots problem (as well as the MinMax versions of the barrier and room problems).

- The coresets presented in Section 5 are for fixed customer weights. Recently, Feldman et al. [13] showed that a weak coreset for the classical MinSum for variable weighted customers exists. Whether their results extend to the collection depots problem is still open.
- We have looked at the collection depots location problem in the rectilinear metric, and preliminary results indicate that the proposed solutions are more efficient in this metric than in the Euclidean metric. Similar improvements were observed by Tamir and Halman [25] for the MinMax collection depots problem. Further investigations are needed.

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